Gravity model for topological features on a cylindrical manifold

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Abstract

A model aimed at understanding quantum gravity in terms of Birkhoff's approach is discussed. The geometry of this model is constructed by using a winding map of Minkowski space into a $\mathbb{R}^3 \times S^1$-cylinder. The basic field of this model is a field of unit vectors defined through the velocity field of a flow wrapping the cylinder. The degeneration of some parts of the flow into circles (topological features) results in inhomogeneities and gives rise to a scalar field, analogous to the gravitational field. The geometry and dynamics of this field are briefly discussed. We treat the intersections between the topological features and the observer's 3-space as matter particles and argue that these entities are likely to possess some quantum properties.

1 Introduction

In this paper we shall discuss a mathematical construction aimed at understanding quantum gravity in terms of Birkhoff's twist Hamiltonian diffeomorphism of a cylinder [1]. We shall also use the idea of compactification of extra dimensions due to Klein [2]. To outline the main idea behind this model in a very simple way, we can reduce the dimensionality and consider the dynamics of a vector field defined on a 2-cylinder $\mathbb{R}^1 \times S^1$. For this purpose we can use the velocity field $u(x, \tau)$ of a two-dimensional flow of ideal incompressible fluid moving through this manifold.

Indeed, the dynamics of the vector field $u(x, \tau)$ with the initial condition $u(x, 0)$ is defined by the evolution equation

$$\delta \int_{\Delta\tau} \int_{\Delta x} dx \wedge u(x, \tau) d\tau \to 0, \tag{1.1}$$

where we use the restriction of the vector field onto an arbitrary cylinder's element; $\Delta\tau$ is the evolution (time) interval, and Δx is an arbitrary segment

of the cylinder's element. In other words, we assume the variation of the integral of the mass carried by the flow through the segment during a finite time interval to be vanishing. That is, as a result of the field evolution, $u(x,0) \to u(x,\infty)$, the functional of the flow mass approaches to its maximal value. If, at the initial moment of time, the regular vector field $u(x,0)$ corresponds to a unit vector forming an angle φ with the cylinder's element, then the evolution of this field is described by the equation

$$\delta \int_{\Delta\tau} \int_{\Delta x} dx \wedge u(x,\tau) d\tau = \delta \int_{\Delta\tau} \int_{\Delta x} \sin\varphi(\tau) dx d\tau = \cos\varphi(\tau) \Delta\tau \Delta x \to 0. \tag{1.2}$$

Therefore, the case of $\varphi(0) = 0$ corresponds to the absolute instability of the vector field. During its evolution, $u(x,0) \to u(x,\infty)$, the field is relatively stable at $0 < \varphi(\tau) < \pi/2$, achieving the absolute stability at the end of this evolution, when $\varphi(\infty) = \pi/2$. If, additionally, we fix the vector field $u(x,\tau)$ at the endpoints of the segment Δx by imposing some boundary conditions on the evolution equation 1.1, we would get the following dynamical equation:

$$\delta \int_{\Delta\tau} \int_{\Delta x} dx \wedge u(x,t) d\tau = 0. \tag{1.3}$$

Let some flow lines of the vector field $u(x,\tau)$ be degenerated into circles (topological features) as a result of the absolute instability of the field and fluctuations during the initial phase of its evolution. Since the dynamics of such topological features is described by 1.3, the features would tend to move towards that side of Δx where the field $u(x,\tau)$ is more stable. Thus, the topological features serve as attraction points for each other and can be used for modelling matter particles (mass points).

We must emphasise that the plane (x,τ), in which our variational equations are defined, has the Euclidean metric. That is, in the case of the Euclidean plane (x,ϕ) wrapping over a cylinder we can identify the azimuthal parameter ϕ with the evolution parameter τ. By choosing the observer's worldline coinciding with a cylinder's element we can speak of a classical limit, whereas by generalising and involving also the azimuthal (angular) parameter we can speak of the quantisation of our model. So, when the observer's worldline is an arbitrary helix on the cylinder, the variational equation 1.3 reads

$$\delta \int_{\Delta x_0} \int_{\Delta x_1} dx_1 \wedge g(x) dx_0 = 0, \tag{1.4}$$

where the varied is the vector field $g(x)$ defined on the pseudo-Euclidean plane (x_0, x_1) oriented in such a way that one of its isotropic lines covers the cylinder-defining circle and the other corresponds to a cylinder's element. In

this case we can speak of a relativistic consideration. If the observer's worldline corresponds to a curved line orthogonal to the flow lines of the vector field $g(x)$, where $g^2(x) > 0$, then we have to use the variational equation defined on a two-dimensional pseudo-Riemann manifold M induced by the vector field $g(x)$, namely,

$$\delta \int_{\Delta M} g^2(x') \sqrt{-\det g_{ij}}\, dx'_0 \wedge dx'_1 = 0, \tag{1.5}$$

where $\Delta M = \Delta x'_0 \times \Delta x'_1$ is an arbitrary region of the manifold M; $x'_0(\phi)$ is the flow line of the vector field $g(x)$ parameterised by the angular coordinate ϕ; $x'_1(r)$ is the spatial coordinate on the cylinder (orthogonal to the observer worldline) parameterised by the Euclidean length r; g_{ij} is the Gram matrix corresponding to the pair of tangent vectors $\left(\frac{dx'_0}{d\phi}, \frac{dx'_1}{dr}\right)$. In this case the dynamics of the vector field is described through the geometry of its flow lines [3, 4, 5].

Thus, we can say that our approach to the dynamics of the vector field is based on maximisation of the mass carried by the flow [6, 7], which is not exactly what is typically used in the ergodic theory [8, 9, 10]. However, this principle is likely to be related to the the minimum principle for the velocity field [13, 14, 15], which is a special case of the more general principle of minimum or maximum entropy production [11, 12].

Before a more detailed discussion of this model we have to make a few preliminary notes. First, throughout this paper we shall use a somewhat unconventional spherical coordinates. Namely, latitude will be measured modulo 2π and longitude – modulo π. In other words, we shall use the following spherical $(\rho,\ \varphi,\ \theta_1, \ldots, \theta_{n-2})$ to Cartesian $(x_1, \ldots, x_n)$ coordinate transformation in $\mathbb{R}^n$:

$$\begin{aligned}
x_1 &= \rho \cos\varphi, \\
x_2 &= \rho \sin\varphi \cos\theta_1, \\
x_3 &= \rho \sin\varphi \sin\theta_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{n-1} &= \rho \sin\varphi\ \ldots\ \sin\theta_{n-3} \cos\theta_{n-2}, \\
x_n &= \rho \sin\varphi\ \ldots\ \sin\theta_{n-3} \sin\theta_{n-2},
\end{aligned}$$

where $0 \le \rho < \infty$, $0 \le \varphi < 2\pi$ and $0 \le \theta_i < \pi$. We shall also be interpreting the projective space RP^n as the space of centrally symmetric lines in $\mathbb{R}^{n+1}$, that is, as a quotient space $\mathbb{R}^{n+1} \backslash \{0\}$ under the equivalence relation $x \sim rx$, where $r \in \mathbb{R} \backslash \{0\}$.

2 The geometry of the model

We can describe the geometry of our model in terms of the mapping of the Euclidean plane into a 2-sphere, S^2, by winding the former around the latter. We can also use similar winding maps for the pseudo-Euclidean plane into a cylinder, $\mathbb{R} \times S^1$, or a torus, $S^1 \times S^1$. More formally this could be expressed in the following way [16]. Take the polar coordinates (φ, ρ) defined on the Euclidean plane and the spherical coordinates (θ, ϕ) on a sphere. We can map the Euclidean plane into sphere by using the congruence classes modulo π and 2π. That is,

$$\theta = |\varphi| \mod \pi, \quad \phi = |\pm \pi\rho| \mod 2\pi, \tag{2.1}$$

where the positive sign corresponds to the interval $0 \leq \varphi < \pi$ and negative – to the interval $\pi \leq \varphi < 2\pi$. If the projective lines are chosen to be centrally symmetric then the Euclidean plane can be generated as the product $RP^1 \times \mathbb{R}$. Here the components of $\mathbb{R}$ are assumed to be Euclidean, i.e., rigid and with no mirror-reflection operation allowed. Similarly, we can define a space based on unoriented lines in the tangent plane to the sphere. Therefore, the sphere can be generated by the product $RP^1 \times S^1$, the opposite points of the circle being identified with each other. In this representation all centrally symmetric Euclidean lines are mapped as

$$\mathbb{R} \to S^1 : e^{i\pi x} = e^{\pm i\pi\rho} \tag{2.2}$$

by winding them onto the corresponding circles of the sphere.

The winding mapping of Euclidean space onto a sphere can be extended to any number of dimensions. Here we are focusing mostly on the case of Euclidean space, $\mathbb{R}^3$, generated as the product $RP^2 \times \mathbb{R}$ and also on the case of a 3-sphere generated as $RP^2 \times S^1$. In both cases we assume the Euclidean rigidity of straight lines and the identification of the opposite points on a circle. Euclidean space, $\mathbb{R}^3$, can be mapped into a sphere, S^3, by the winding transformation analogous to 2.1. Indeed, for this purpose we only have to establish a relation between the length of the radius-vector in Euclidean space and the spherical coordinate (latitude) measured modulo 2π. The relevant transformations are as follows:

$$\theta_1 = \vartheta, \quad \theta_2 = |\varphi| \mod \pi, \quad \phi = |\pm \pi\rho| \mod 2\pi, \tag{2.3}$$

where the sign is determined by the quadrant of φ.

Let (e_0, e_1) be an orthonormal basis on a pseudo-Euclidean plane with coordinates (x_0, x_1). Let the cylindrical coordinates of $\mathbb{R} \times S^1$ be (ϕ, r).

Then the simplest mapping of this pseudo-Euclidean plane to the cylinder would be

$$\phi = |\pi(x_0 + x_1)| \mod 2\pi, r = x_0 - x_1. \tag{2.4}$$

That is, the first isotropic line is winded here around the cylinder's cross-section (circle) and the second line is identified with the cylinder's element. In this way one can make a correspondence between any non-isotropic (having a non-zero length) vector in the plane and a point on the cylinder. For instance, if a vector x having coordinates (x_0, x_1) forms a hyperbolic angle φ with the e_0 or $-e_0$, then

$$\phi = | \pm \pi e^{-\varphi}\rho| \mod 2\pi = |\pi(x_0 + x_1)| \mod 2\pi. \tag{2.5}$$

If this vector forms the hyperbolic angle φ with the e_1 or $-e_1$, then

$$r = \pm e^{\varphi}\rho = x_0 - x_1, \tag{2.6}$$

where $\varphi = -\ln\left|\frac{x_0+x_1}{\rho}\right|$; $\rho = |(x_0 + x_1)(x_0 - x_1)|^{1/2}$.

By analogy, one can build a winding map of the pseudo-Euclidean plane into the torus, with the only difference that in the latter case the second isotropic line is winded around the longitudinal (toroidal) direction of the torus.

Now let us consider a 6-dimensional pseudo-Euclidean space $\mathbb{R}^6$ with the signature $(+, +, +, -, -, -)$. In this case the analogue to the cylinder above is the product $\mathbb{R}^3 \times S^3$, in which the component $\mathbb{R}^3$ is Euclidean space. In order to wind the space $\mathbb{R}^6$ over the cylinder $\mathbb{R}^3 \times S^3$ we have to take an arbitrary pseudo-Euclidean plane in $\mathbb{R}^6$ passing through the (arbitrary) orthogonal lines x_k, x_p that belong to two Euclidean subspaces $\mathbb{R}^3$ of the space $\mathbb{R}^6$. Each plane (x_k, x_p) has to be winded onto a cylinder with the cylindrical coordinates (ϕ_k, r_p); the indices k, p correspond to the projective space RP^2. We can take all the possible planes and wind them over the corresponding cylinders. The mapping transformation of the pseudo-Euclidean space $\mathbb{R}^6$ into the cylinder $\mathbb{R}^3 \times S^3$ is similar to the expressions 2.5 and 2.6:

$$\phi_k = | \pm \pi e^{-\varphi}\rho| \mod 2\pi = |\pi(x_k + x_p)| \mod 2\pi, \tag{2.7}$$

$$r_p = \pm e^{\varphi}\rho = x_k - x_p. \tag{2.8}$$

By fixing the running index k and replacing it with zero we can get the winding map of the Minkowski space $\mathbb{R}^4$ into the cylinder $\mathbb{R}^3 \times S^1$, which is a particular case (reduction) of 2.7 and 2.8. Conversely, by winding $\mathbb{R}^3$ over a 3-sphere, S^3, we can generalise the case and derive a winding map from $\mathbb{R}^6$ into $S^3 \times S^3$.

Let us consider the relationship between different orthonormal bases in the pseudo-Euclidean plane, which is winded over a cylinder. It is known that all of the orthonormal bases in a pseudo-Euclidean are equivalent (i.e., none of them can be chosen as privileged). However, by defining a regular field c of unit vectors on the pseudo-Euclidean plane it is, indeed, possible to get such a privileged orthonormal basis (c, c_1). In turn, a non-uniform unitary vector field $g(x)$, having a hyperbolic angle $\varphi(x)$ with respect to the field c, would induce a non-orthonormal frame $(g'(x), g_1'(x))$. Indeed, if we assume that the following equalities are satisfied:

$$\pi = | \pm \pi e^{-\varphi} \rho(e^{\varphi} g)| \mod 2\pi = | \pm \pi e^{-\varphi} \rho(g')| \mod 2\pi, \tag{2.9}$$

$$\pm 1 = \pm e^{\varphi} \rho(e^{-\varphi} g_1) = \pm e^{\varphi} \rho(g_1'), \tag{2.10}$$

we can derive a non-orthonormal frame $(g'(x), g_1'(x))$ by using the following transformation of the orthonormal frame $(g(x), g_1(x))$:

$$g'(x) = e^{\varphi} g(x), \quad g_1'(x) = e^{-\varphi} g_1(x). \tag{2.11}$$

Then the field $g(x)$ would induce a 2-dimensional pseudo-Riemann manifold with a metric tensor $\{g_{ij}'\}$ (where $i, j = 0, 1$), which is the same as the Gram matrix corresponding to the system of vectors $(g'(x), g_1'(x))$. A unitary vector field $g(x)$ defined in the Minkowski space winded onto the cylinder $\mathbb{R}^3 \times S^1$ would induce a 4-dimensional pseudo-Riemann manifold. Indeed, take the orthonormal frame (g, g_1, g_2, g_3) derived by hyperbolically rotating the Minkowski space by the angle $\varphi(x)$ in the plane $(g(x), c)$. Then the Gram matrix g_{ij}' $(i, j = 0, 1, 2, 3)$ corresponding to the set of vectors $\{e^{\varphi} g, e^{-\varphi} g_1, g_2, g_3\}$ would be related to the metric of the pseudo-Riemann manifold. Note, that, since the determinant of the Gram matrix is unity [17, 18], the induced metric preserves the volume. That is, the differential volume element of our manifold is equal to the corresponding volume element of the Minkowski space.

3 The dynamics of the model

As we have already mentioned in Section 1, the dynamics of the velocity field $u(x, \tau)$ of an ideal incompressible fluid on the surface of a cylinder $\mathbb{R}^3 \times S^1$ can be characterised by using the minimal volume principle, i.e., by assuming that the 4-volume of the flow through an arbitrary 3-surface $\Sigma \subset \mathbb{R}^3$ during the time T is minimal under some initial and boundary conditions, namely:

$$\delta \int_0^T \int_\Sigma dV \wedge u(x, \tau) d\tau = 0, \tag{3.1}$$

where dV is the differential volume element of a 3-surface Σ. This is also equivalent to the minimal mass carried by the flow through the measuring surface during a finite time interval.

In a classical approximation, by using the winding projection of the Minkowski space into a cylinder $\mathbb{R}^3 \times S^1$, we can pass from the dynamics defined on a cylinder to the statics in the Minkowski space. Let the global time t be parameterised by the length of the flow line of the vector field c in the Minkowski space corresponding to some regular vector field on the cylinder and let the length of a single turn around the cylinder be h. Let us take in the Minkowski space a set of orthogonal to c Euclidean spaces $\mathbb{R}^3$ in the Minkowski space. The distance between these spaces is equal to hz, where $z \in \mathbb{Z}$. The projection of this set of spaces into the cylinder is a three-dimensional manifold, which we shall refer to as a global measuring surface. Then we can make a one-to-one correspondence between the dynamical vector field $u(x, \tau)$ and the static vector field $g(x)$, defined in the Minkowsky space. Thus, in a classical approximation there exists a correspondence between the minimisation of the 4-volume of the flow $u(x, \tau)$ on the cylinder and the minimisation of the 4-volume of the static flow defined in the Minkowski space by the vector field $g(x)$, namely:

$$\delta \int_0^{x_0} \int_{\Sigma'} dV \wedge g(x) dx_0 = 0, \tag{3.2}$$

where the first basis vector e_0 coincides with the vector c, and the 3-surfaces, Σ', lie in the Euclidean sub-spaces orthogonal to the vector c. Let $\{(c_i\}) = (c_0, c_1, c_2, c_3)$ be an orthonormal basis in $\mathbb{R}^4$ such that $c_0 = c$. Let the reference frame bundle be such that each non-singular point of $\mathbb{R}^4$ has a corresponding non-orthonormal frame $(g_i(x)) = (g_0, g_1, g_2, g_3)$, where $g_0 = g(x)$, $g_1 = c_1$, $g_2 = c_2$, $g_3 = c_3$. Let us form a matrix $\{g_{ij}\}$ of inner products (c_i, g_j) of the basis vectors $\{c_i\}$ and the frame $\{g_i\}$. The absolute value of its determinant, $\det(g_{ij})$, is equal to the volume of the parallelepiped formed by the vectors (g_0, g_1, g_2, g_3). It is also equal to the scalar product, $(g(x), c)$. On the other hand, the equation $(g(x), c)^2 = |\det G(x)|$ holds for the Gram matrix, $G(x)$, which corresponds to the set of vectors $\{g_i(x)\}$ [21]. Then, according to the principle 3.2, the vector field $g(x)$ satisfies the variational equation

$$\delta \int_\Omega (g(x), c) dx^4 = \delta \int_\Omega |\det G(x)|^{\frac{1}{2}} dx^4 = 0, \tag{3.3}$$

where dx^4 is the differential volume element of a cylindrical 4-region Ω of the Minkowski space, having the height T. The cylinder's base is a 3-surface Σ with the boundary condition $g(x) = c$. In order to derive the differential equation satisfying the integral variational equation 3.3, we have to find

the elementary region of integration, Ω. Let $\Delta\pi$ be an infinitesimal parallelepiped spanned by the vectors $\Delta x_0, \Delta x_1, \Delta x_2, \Delta x_3$, with ω being a tubular neighbourhood with the base spanned by the vectors $\Delta x_1, \Delta x_2, \Delta x_3$. This (vector) tubular neighbourhood is filled in with the vectors $|\Delta x_0| g(x)$ obtained from the flow lines of the vector field $g(x)$ by increasing the natural parameter (the pseudo-Euclidean length) by the amount $|\Delta x_0|$. Then the localisation expression of the equation 3.3 gives [19]:

$$\delta \int_{\Delta\pi} |\det G(x,t)|^{\frac{1}{2}} dx^4 = \delta \operatorname{Vol} \omega = 0. \tag{3.4}$$

Since the field lines of a nonholonomy vector field $g(x)$ are nonparallel even locally, any variation of such a field (i.e, the increase or decrease of its nonholonomicity) would result in a non-vanishing variation of the volume $\operatorname{Vol}\omega$. Conversely, in the case of a holonomy field its variations do not affect the local parallelism, so that the holonomicity of the field $g(x)$ appears to be the necessary condition for the zero variation of $\operatorname{Vol}\omega$. Given a vector field $g(x)$ with an arbitrary absolute value, the sufficient conditions for the vanishing variation of the volume of the tubular neighbourhood ω are the potentiality of this field and the harmonic character of its potential. In terms of differential forms these conditions correspond to a simple differential equation:

$$d \star g(x) = 0, \tag{3.5}$$

Where d is the external differential; $\star$ is the Hodge star operator; $g(x) = d\varphi(x)$; and $\varphi(x)$ is an arbitrary continuous and smooth function defined everywhere in the Minkowski space, except for the singularity points (topological features). Substituting the unitary holonomy field $g(x) = k(x)d\varphi(x)$ in 3.5, where $k(x) = 1/|d\varphi(x)|$, we shall find that the unitary vector field $g(x)$ must satisfy the minimum condition for the integral surfaces of the covector field dual to $g(x)$. In this case the magnitude of the scalar quantity $\varphi(x)$ will be equal to the hyperbolic angle between the vectors $g(x)$ and c. We can also note that the potential vector field $g(x) = d\varphi(x)$ represented by the harmonic functions $\varphi(x)$ is the solution to the following variational equation:

$$\delta \int_0^T \int_\Sigma \left[\left(\frac{\partial \varphi(x,t)}{\partial t} \right)^2 - \nabla^2 \varphi(x,t) \right] dx^3 dt = 0, \tag{3.6}$$

in which Σ is a region in Euclidean space of the "global" observer; the function $\varphi(x,t)$ is defined in the Minkowski space. Thus, the stationary scalar field $\varphi(x)$ induced by a topological feature in the global space is identical to the Newtonian gravitational potential of a mass point.

We have to bear in mind that the space of a "real" observer is curved, since the line for measuring time and the surface for measuring the flux is defined by the vector field $g(x)$, and not by the field c as in the case of the global observer. Therefore, if we wish to derive a variational equation corresponding to the real observer, we have to define it on the pseudo-Riemann manifold M induced in the Minkowski space by the holonomy field $g(x)$, whose flux is measured through the surfaces orthogonal to its flow lines and whose flow lines serve for measuring time. The metric on M is given by the Gram matrix of four tangent vectors, one of which corresponds to the flow line $x'_0(\phi)$ parameterised by the angular coordinate of the cylindrical manifold, and the three others are tangent to the coordinate lines of the 3-surface $x'_1(r), x'_2(r), x'_3(r)$ parameterised by the Euclidean length. The following variational equation holds for an arbitrary region ΔM of M:

$$\delta \int_{\Delta M} g^2(x')dV = 0 \tag{3.7}$$

(under the given boundary conditions) where dV is the differential volume element of M. Note that the norm of the vector $g(x)$ coincides with the magnitude of the volume-element deformation of the pseudo-Riemann manifold, which allows making the correspondence between our functional and that of the Hilbert-Einstein action.

Returning to the global space, let us consider some properties of the vector field $g(x)$. Let a point in the Minkowski space has a trajectory $X(\tau)$ and velocity $\dot{X}$. Its dynamics is determined by the variational equation:

$$\delta \int_0^T (g(x), \dot{X})d\tau = 0. \tag{3.8}$$

The varied here is the trajectory $X(\tau)$ in the Minkowski space where the vector field $g(x)$ is defined and where the absolute time τ plays the role of the evolution parameter. For small time intervals the integral equation 3.8 can be reduced to

$$\delta(g(x), \dot{X}) = 0, \tag{3.9}$$

which is satisfied by the differential equation

$$\ddot{X} = g(X). \tag{3.10}$$

Taking the orthogonal projection $\xi(\tau) = \mathrm{pr}_{\mathbb{R}^3} X(\tau)$ of the trajectory of a given topological feature in Euclidean space of the global observer, as well as the projection $\nabla\varphi(X) = \mathrm{pr}_{\mathbb{R}^3} g(X)$ of the vector field $g(x)$ at the point $X(\tau)$ gives a simple differential equation

$$\ddot{\xi}(\tau) = \nabla\varphi(x), \tag{3.11}$$

which (as in Newtonian mechanics) expresses the fact that the acceleration of a mass point in an external gravitational field does not depend on the mass.

4 Some implications

Let us consider some implications of our model for a real observer in a classical approximation (by the real observer we mean the reference frame of a topological feature). First, we can note that a real observer moving uniformly along a straight line in the Minkowski space cannot detect the "relative vacuum" determined by the vector c and, hence, cannot measure the global time t. By measuring the velocities of topological features (also uniformly moving along straight lines) our observer would find that for gauging space and time one can use an arbitrary unitary vector field c' defined on the Minkowski space. Therefore, the observer would conclude that the notion of spacetime should be relative. It is seen that the real observer can neither detect the unitary vector field $g(x)$ nor its deviations from the vector c. However, it would be possible to measure the gradient of the scalar (gravitational) field and detect the pseudo-Riemann manifold induced by $g(x)$.

Indeed, in order to gauge time and distances in different points of space (with different magnitudes of the scalar field) one has to use the locally orthonormal basis $\{g'_i\}$ defined on the 4-dimensional pseudo-Riemann manifold with its metric tensor $\{g'_{ij}\}$. Thus, for the real observer, the deformations of the pseudo-Euclidean space could be regarded as if induced by the scalar field. Locally, the deformations could be cancelled by properly accelerating the mass point (topological feature), which implies that its trajectory corresponds to a geodesics of the manifold.

We can see that the dynamics of a topological feature in our model is identical to the dynamics of a mass point in the gravitational field. Indeed, the scalar field around a topological feature is spherically symmetric. At distance r from the origin the metric will be $e^{2\varphi}dt^2 - e^{-2\varphi}dr^2$, which corresponds to the metric tensor of the gravitational field of a point mass, given $e^{2\varphi} \approx 1 + 2\varphi$ for small φ. If $\varphi = H\tau$, i.e., hyperbolic angle φ linearly depends from the evolutionary parameter τ, then we can compare the constant H with the cosmological factor.

Let us now consider some quantum properties of our model. Let the absolute value of the vector field c be a continuous function $|c(x)|$ in the Minkowski space. Then the angular velocity of the flow will be:

$$\dot{\phi}(x) = \frac{d\phi(x)}{dt} = \frac{\pi}{h}|c(x)|, \tag{4.1}$$

where the angular function $\phi(x)$ can be identified with the phase action of the gauge potential in the observer space. On the other hand, it is reasonable to associate the angular velocity $X(\tau)$ of the topological feature with the Lagrangian of a point mass in the Minkowski space:

$$\dot{\phi}(X) = \frac{d\phi(X)}{d\tau} = \frac{\pi}{h}L(x). \tag{4.2}$$

Let us consider the random walk process of the topological feature in the cylinder space $\mathbb{R}^3 \times S^1$. Let a probability density function $\rho(x)$ be defined on a line, such that $\rho(x)$,

$$\int_{-\infty}^{+\infty} \rho(x)dx = 1. \tag{4.3}$$

Let us calculate the expectation value for the random variable $e^{i\pi x}$, which arises when a line is compactified into a circe:

$$M(e^{i\pi x}) = \int_{-\infty}^{+\infty} \rho(e^{i\pi x})dx = \int_{-\infty}^{+\infty} e^{i\pi x}\rho(x)dx = pe^{i\pi\alpha}. \tag{4.4}$$

Here the quantity $pe^{i\pi\alpha}$ can be called the complex probability amplitude. It characterises two parameters of the random variable distribution, namely, the expectation value itself, $e^{i\pi\alpha}$, and the probability density, p, i.e. the magnitude of the expectation value. If $\rho(x) = \delta(\alpha)$, then $M(e^{i\pi x}) = 1 \cdot e^{i\pi\alpha}$. Conversely, if $\rho(x)$ is uniformly distributed along the line then the expectation value is $M(e^{i\pi x}) = 0$. It follows from these considerations that a distribution in $\mathbb{R}^3$ of a complex probability amplitude is related to random events in the cylinder space $\mathbb{R}^3 \times S^1$.

In order to specify the trajectories $X(\tau)$ in the Minkowski space with an external angular potential $\phi(x)$ we shall use the procedure proposed by Feynman [22]. Let the probabilistic behaviour of the topological feature be described as a Markov random walk in the cylinder space $\mathbb{R}^3 \times S^1$. An elementary event in this space is a free passage. In the Minkowski space such an event is characterised by two random variables, duration, $\Delta\tau$, and the random path vector, ΔX, whose projection into Euclidean space of the absolute observer is $\Delta\xi$. The ratio $\frac{\Delta\xi}{\Delta\tau}$ is a random velocity vector, $\dot{\xi}$. On the other hand, the free passage of a topological feature corresponds to an increment in the phase angle $\Delta\phi(X) = \dot{\phi}(X)\Delta\tau$ (phase action) in the cylinder space $\mathbb{R}^3 \times S^1$.

Let the probability distribution of the phase action has an exponential form, say, $\rho(\Delta\phi) = e^{-\Delta\phi}$ (neglecting the normalisation coefficient). Then, the corresponding probability density for the random variable $e^{i\Delta\phi}$ will be

$$\rho(e^{i\Delta\phi}) = e^{-\Delta\phi}e^{i\Delta\phi}. \tag{4.5}$$

Using the properties of a Markov chain [20], we can derive the probability density for an arbitrary number of random walks:

$$\rho(e^{i\phi}) = \prod_0^T e^{-\dot{\phi}d\tau} e^{i\dot{\phi}d\tau}. \tag{4.6}$$

To get the expectation value of the random variable $e^{i\phi}$ we have to sum up over the all possible trajectories, that is, to calculate the quantity

$$M(e^{i\phi}) = \sum \prod_0^T e^{-\dot{\phi}d\tau} e^{i\dot{\phi}d\tau}. \tag{4.7}$$

It is known that any non-vanishing variation of the phase action has a vanishing amplitude of the transitional probability and, on the contrary, that the vanishing variation corresponds to a non-vanishing probability amplitude [23, 24, 25]. Then it is seen that the integral action corresponding to the topological feature must be minimal. It follows that the "probabilistic trap" of a random walk [26] in the cylinder space $\mathbb{R}^3 \times S^1$ is determined by the variational principle – the same that determines the dynamics of a mass point in classical mechanics.

5 Conclusions

In conclusion, we have made an attempt to describe the dynamics of space-time (as well as of matter particles) in terms of the vector field defined on a cylindrical manifold and based on the principle of maximum mass carried by the field flow. The analysis of the observational implications of our model sheds new light on the conceptual problems of quantum gravity.

Still many details of our model are left unexplored. For example, it would be instructive to devise the relationship between the vector field $g(x)$ and the 4-potential of electromagnetic field $A(x)$ and to consider the local perturbations of $g(x)$ as gravitons or/and photons. We also expect that the most important properties of our model would be revealed by extending it to the cylindrical manifold $\mathbb{R}^3 \times S^3$. In particular, we hope that within such an extended version of our framework it would be possible to find a geometric interpretation of all known gauge fields. It is also expected that studying the dynamics of the minimal unit vector field on a 7-sphere should be interesting for cosmological applications of our approach.

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